CMS, resit Test 2

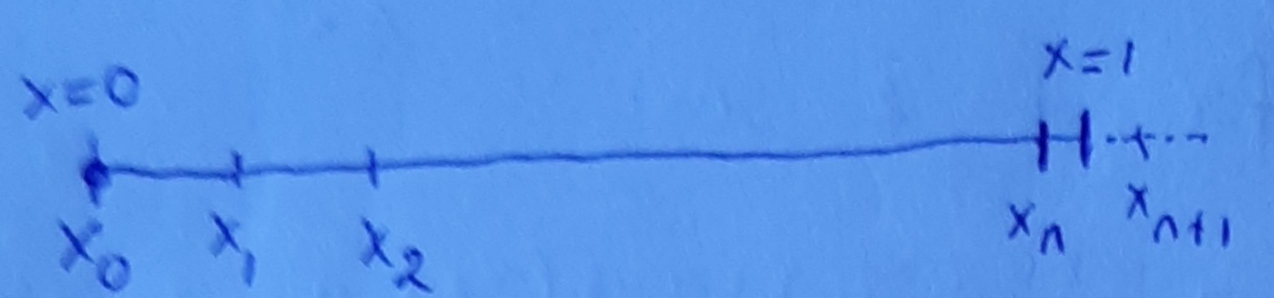
1. on $[0,1]$ $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\cos(x)\frac{\partial u}{\partial x}\right)$ for $t>0$

$u(0,t)=0 \quad u_x(1,t)=0$ boundary cond.

$u(x,0) = \sin(\frac{\pi}{2}x) \quad u_t(x,0)=0$ initial cond.

a. second-order accurate discretization in space on an equidistant grid with grid size Δx
left boundary condition at grid point, right in middle between two grid points

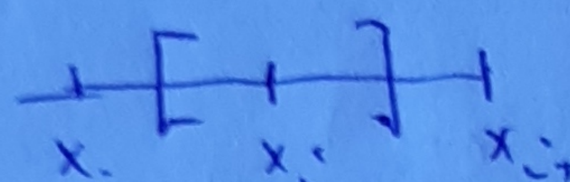
0.4 • Grid:



$(n+\frac{1}{2})\Delta x = 1 \Rightarrow \Delta x = \frac{1}{n+\frac{1}{2}}$

$x_i = i\Delta x \qquad i=0,\dots,n$

1.2 • Finite volume



volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial^2 u}{\partial t^2}\,dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{\partial}{\partial x}\cos(x)\frac{\partial u}{\partial x}\right)dx$$

Note: $u(x,t)$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(t,x)\,dx = \cos x \frac{\partial u}{\partial x}\Big|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}$$

$$= \cos x_{i+\frac{1}{2}} \frac{\partial u}{\partial x}\Big|_{x_{i+\frac{1}{2}}} - \cos x_{i-\frac{1}{2}} \frac{\partial u}{\partial x}\Big|_{x_{i-\frac{1}{2}}}$$

• approximate (second order approximation)

$$\Rightarrow \frac{d^2}{dt^2} u_i \Delta x = \cos x_{i+\frac{1}{2}} \frac{u_{i+1}-u_i}{\Delta x} - \cos x_{i-\frac{1}{2}} \frac{u_i - u_{i-1}}{\Delta x},$$

with $u_i = u_i(t) \approx u(x_i, t)$

holds for $i = 1, \dots, n-1$

0.5 using $u_0 = 0$ (i.e. $u_0(t)=0$)

• at $i=n$ use Neumann Boundary condition,

$$u_x(1,t) = 0 \Rightarrow \frac{\partial u}{\partial x}\Big|_{x_{n+\frac{1}{2}}} \approx 0 \Rightarrow \frac{u_{n+1}-u_n}{\Delta x} = 0$$

$\Rightarrow u_{n+1} = u_n$

$$\Rightarrow \frac{d^2}{dt^2} u_n \Delta x = -\cos_{n-\frac{1}{2}} \frac{u_n - u_{n-1}}{\Delta x}$$

0.4 • initial conditions

$u_i(0) = \sin(\frac{\pi}{2}x_i) \qquad i=1,\dots,n$

$\frac{du_i}{dt}(0) = 0$

b. $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $A = \begin{bmatrix} -\frac{\cos(x_{1\frac{1}{2}})+\cos(x_{\frac{1}{2}})}{\Delta x^2} & \frac{\cos(x_{1\frac{1}{2}})}{\Delta x^2} & & \\ \frac{\cos(x_{1\frac{1}{2}})}{\Delta x^2} & \ddots & \frac{\cos(x_{2\frac{1}{2}})}{\Delta x^2} & \\ & \ddots & \ddots & \\ & \frac{\cos(x_{n-\frac{3}{2}})}{\Delta x^2} & -\frac{\cos(x_{n-\frac{1}{2}})+\cos(x_{n-\frac{3}{2}})}{\Delta x^2} & \frac{\cos(x_{n-\frac{1}{2}})}{\Delta x^2} \\ & & \frac{\cos(x_{n-\frac{1}{2}})}{\Delta x^2} & -\frac{\cos(x_{n-\frac{1}{2}})}{\Delta x^2} \end{bmatrix}$

since $\frac{d^2u_i}{dt^2} = \frac{1}{\Delta x}\left[\cos(x_{i+\frac{1}{2}})\frac{u_{i+1}-u_i}{\Delta x} - \cos(x_{i-\frac{1}{2}})\frac{u_i-u_{i-1}}{\Delta x}\right]$ $i=1,..,n-1$

0.5 $\Rightarrow$ • $\frac{d^2u_i}{dt^2} = \frac{\cos(x_{i-\frac{1}{2}})}{\Delta x^2}u_{i-1} - \frac{\cos(x_{i+\frac{1}{2}})+\cos(x_{i-\frac{1}{2}})}{\Delta x^2}u_i + \frac{\cos(x_{i+\frac{1}{2}})}{\Delta x^2}u_{i+1}$

for $i = 2,\dots,n-1$

• $\frac{d^2u_1}{dt^2} = -\frac{\cos(x_{1\frac{1}{2}})+\cos(x_{\frac{1}{2}})}{\Delta x^2}u_1 + \frac{\cos(x_{1\frac{1}{2}})}{\Delta x^2}u_2$

and

• $\frac{d^2u_n}{dt^2} = +\frac{\cos(x_{n-\frac{1}{2}})}{\Delta x^2}u_{n-1} - \frac{\cos(x_{n-\frac{1}{2}})}{\Delta x^2}u_n$

0.2 • A is symmetric (see matrix above)

• initial conditions, $t=0$

$\vec{u}(0) = \begin{pmatrix} \sin\frac{\pi}{2}x_1 \\ \sin\frac{\pi}{2}x_2 \\ \vdots \\ \sin\frac{\pi}{2}x_n \end{pmatrix}$ $\frac{d\vec{u}}{dt}(0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

0.3

c. difference / Fourier method

Typical row of A

$L_{\Delta x}u_j = \frac{\cos(x_{j-\frac{1}{2}})}{\Delta x^2}u_{j-1} - \frac{\cos(x_{j+\frac{1}{2}})+\cos(j-\frac{1}{2})}{\Delta x^2}u_j + \frac{\cos(x_j)}{\Delta x^2}u_{j+1}$

0.3 set $u_j = e^{ij\theta}$, $i=\sqrt{-1}$

• coefficient are largest near $x=0$, so we take $x=0$ and then the coefficients are $\frac{1}{\Delta x^2}, -\frac{2}{\Delta x^2}, \frac{1}{\Delta x^2}$

0.4 so consider $L_{\Delta x}u_j = \frac{1}{\Delta x^2}(u_{j-1} - 2u_j + u_{j-1})$

$L_{\Delta x}u_j = \lambda u_j$

$$L_{\Delta x} u_j \underset{\uparrow}{=} \frac{1}{\Delta x^2}\left(e^{i(j-1)\theta} - 2e^{ij\theta} + e^{i(j+1)\theta}\right)$$

$u_j = e^{ij\theta}$

$$= \frac{1}{\Delta x^2}\left(e^{-i\theta} - 2 + e^{i\theta}\right) e^{ij\theta} = \frac{1}{\Delta x^2}\left(e^{-i\theta} - 2 + e^{i\theta}\right) u_j$$

0.8

$\Rightarrow$ estimate of eigenvalue

$$\lambda = \frac{1}{\Delta x^2}\left(e^{-i\theta} - 2 + e^{i\theta}\right) = \frac{1}{\Delta x^2}\left(2\cos\theta - 2\right) = \frac{-4}{\Delta x^2} \sin^2\frac{\theta}{2}$$

d. $$\frac{d^2\vec{u}}{dt^2} = A\vec{u}$$

define $\vec{v} = \frac{d\vec{u}}{dt} \Rightarrow \frac{d\vec{v}}{dt} = \frac{d^2\vec{u}}{dt^2} = A\vec{u}$

$$\Rightarrow w = \begin{bmatrix} \vec{u} \\ \vec{v} \end{bmatrix}, \quad \frac{dw}{dt} = \begin{bmatrix} \frac{d\vec{u}}{dt} \\ \frac{d\vec{v}}{dt} \end{bmatrix} = \begin{bmatrix} \vec{v} \\ A\vec{u} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{v} \end{bmatrix}$$

0.7

$$\Rightarrow \frac{d}{dt} w = Bw \quad \text{with} \quad B = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

$$w(0) = \begin{bmatrix} \vec{u}(0) \\ \vec{v}(0) \end{bmatrix} = \begin{bmatrix} \vec{u}(0) \\ \frac{d\vec{u}}{dt}(0) \end{bmatrix} = \begin{bmatrix} \sin(\frac{\pi}{2}x_1) \\ \vdots \\ \sin(\frac{\pi}{2}x_n) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

0.3

e. assume λ eigenvalue of matrix A, $-\alpha \leq \lambda \leq 0$ $\quad \alpha > 0$

(i) • according to c) : $\lambda \approx -\frac{4}{\Delta x^2}\sin^2\frac{\theta}{2} \Rightarrow -\frac{4}{\Delta x^2} \leq \lambda \leq 0$

0.5 $\Rightarrow \alpha \approx \frac{4}{\Delta x^2}$

• $$\det(B - \mu I) = \det\begin{bmatrix} -\mu I & I \\ A & -\mu I \end{bmatrix} = \det\begin{bmatrix} -I & \frac{1}{\mu} I \\ A & -\mu I \end{bmatrix} = 0$$

1.0 $$\Rightarrow \det\begin{bmatrix} -I & \frac{1}{\mu} I \\ 0 & -\mu I + \frac{1}{\mu} A \end{bmatrix} = 0 \Rightarrow \det\left(-\mu I + \frac{1}{\mu}A\right) = 0$$

($\nearrow$ 2nd row block + A 1st row block)

$$\Rightarrow \det(A - \mu^2 I) = 0$$

$$\Rightarrow \mu^2 = \lambda(A)$$

but $\begin{cases} \lambda(A) \text{ real, } A \text{ symm.} \\ \lambda(A) < 0 \end{cases}$ (see part c) $\Rightarrow \mu^2 < 0 \Rightarrow \mu$ pure imaginary, $\mu = \pm\sqrt{\lambda}$

$-4/\Delta x^2 \leq \lambda \leq 0 \Rightarrow i\frac{2}{\Delta x} \leq \mu \leq i\frac{2}{\Delta x}$

(ii) the eigenvalue problem originates from substituting a solution
$\vec{w} = \vec{z} e^{\mu t}$ in (2)

this will give the eigenvectors $\vec{z}_k$ and μ_k $k = 1, \dots, 2n$

Hence, the general solution of (2) can be written as

$$\vec{w}(t) = \sum_{k=1}^{2n} \alpha_k \vec{z}_k e^{\mu_k t}$$

where α_k follows from the initial condition

$$\vec{w}(0) = \sum_{k=1}^{\hat{}} \alpha_k \vec{z}$$

Since μ_k are purely imaginary $w(t)$ will not go to 0 for $t \to \infty$

(f) $y' = f(t, y)$ $w_{n+1} = w_{n-1} + 2\Delta t f(t_n, w_n)$

has amplification factor $|\rho(z)| = 1$ for $z = iy$, y real, $y \in [-1, 1]$

- Such a method suitable for our problem:
  no damping in solution of (2) (see ie (ii))
  $\Rightarrow w(t) \not\to 0$ for $t \to \infty$
  hence we wish for no damping in numerical method as well

- we need to have $z = \mu \Delta t \in [-i, i]$, since $|\rho(z)| = 1$, $z = iy$, $y \in [-1, 1]$

$\mu = \pm\sqrt{\lambda}$ $\quad -\alpha \leq \lambda \leq 0 \quad \Rightarrow \quad -i\sqrt{\alpha} \leq \mu \leq i\sqrt{\alpha}$

$\alpha = \frac{4}{\Delta x^2}$

$$\Rightarrow -\frac{2}{\Delta x} i \leq \mu \leq \frac{2}{\Delta x} i$$

$$\Rightarrow -\frac{2}{\Delta x} \Delta t \, i \leq \mu \Delta t \leq \frac{2}{\Delta x} \Delta t \, i$$

$$\Rightarrow 2 \frac{\Delta t}{\Delta x} \leq 1 \quad \Rightarrow \quad \Delta t \leq \frac{\Delta x}{2}$$